A SELF – LEARNING MODULE ON CALCULUS
FOR SCIENCE HIGH SCHOOL

by

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Table of Contents

1  MODULE ON LIMITS AND CONTINUITY: ITS BACKGROUND  
   1.1  Introduction  
   1.2  Background & Rationale  
   1.3  Assumption  
   1.4  Motivation  

2  MODULE PROPER  
   2.1  An Intuitive Concept of the Limit of the Function  
      2.1.1  Specific Objectives  
      2.1.2  Prerequisite Skills  
      2.1.3  Materials Needed  
      2.1.4  Time Frame  
      2.1.5  Lesson Proper  
      2.1.6  Self – Test 2.1  
      2.1.7  Exercise 2.1  
   2.2  An Introduction to and Formal Definition of the Limit of the Function (Graphical Approach)  
      2.2.1  Specific Objectives  
      2.2.2  Prerequisite Skills  
      2.2.3  Materials Needed  
      2.2.4  Time Frame  
      2.2.5  Lesson Proper  
         2.2.5.1  An Introduction to the Limit of the Function  
         2.2.5.2  Limit: The Formal Definition  
      2.2.6  Self – Test 2.2  
      2.2.7  Exercise 2.2  
   2.3  Calculating Limits Using Limit Laws  
      2.3.1  Specific Objectives  
      2.3.2  Prerequisite Skills
2.3.3 Lesson Proper
2.3.4 Self – Test 2.3.A
2.3.5 Self – Test 2.3.B
2.3.6 Exercise 2.3

2.4 One – Sided Limit
2.4.1 Specific Objectives
2.4.2 Prerequisite Skills
2.4.3 Time Frame
2.4.4 Lesson Proper
2.4.5 Self – Test 2.4
2.4.6 Exercise 2.4

2.5 Continuity
2.5.1 Specific Objectives
2.5.2 Prerequisite Skills
2.5.3 Time Frame
2.5.4 Lesson Proper
2.5.5 Self – Test 2.5
2.5.6 Exercise 2.5

3 REFERENCES

4 Answers to Selected Odd-Numbered Self – Test and Exercise Problems
1.1 INTRODUCTION

In the study of calculus, the first important concept or idea that must be introduced is the concept of limit. The limit of a function is the cornerstone of both differential and integral calculus. It is one of the fundamental ideas that distinguishes calculus from other areas of mathematics like algebra or trigonometry.

The main purpose throughout the writing of this module is to present the limit concept and applications to the fourth year high school students in a simple and understandable language with a collection of exercises and carefully solved problems.

This module is designed as user-friendly and contextualized to the needs and level of understanding of its readers.

1.2 BACKGROUND AND RATIONALE

The concept of a limit is very important in the study of calculus that you should understand as very well as possible, and that is easy to do.

This self-learning module will be found helpful to all fourth year students especially those who will go in the fields of science, business and engineering. This consists of limits and continuity involving algebraic functions. We will discuss here
intensively the concept of a limit, its properties and operations, and continuity of the functions.

Specific objectives at the beginning of each lesson are provided so that the reader could have a full grasp of the entire lesson’s activity. A set of exercises at the end of every lesson is given to test the student’s understanding of the concept and problem-solving technique. Each lesson contains carefully chosen examples to facilitate student’s learning and understanding. Several application problems or real-life problems are also included for the diversity in learning. A self-test is also provided at the end of every lesson to help students retain their learning, concepts and skills gained.

Calculus is a mathematical theory, which is based primarily on the concept of a limit. It is believed that this module will meet the needs and intellect of the students. Therefore, it suffices to give all of these concepts to the fourth year high school students. This module will also serve as an effective instructional material.

I believe that mathematics education begins with the pedagogy of mathematics instruction and learning activities contextualized to the needs, interests and intellect of the learners.

1.3 ASSUMPTION

There are certain required knowledge and skills that you should know before you go over with this module. I will therefore assume that you already know the fundamental concepts in advanced algebra and trigonometry in your Pre-Calculus courses. Basic algebraic computational skills have to be studied first so that it will be easier for you to
follow the discussion. You must also have known the properties of inequalities, solving and proving inequalities.

1.4 MOTIVATION

Have you ever wondered how one comes to understand the difficult concept of calculus? Have you ever wanted to know this concept that leads you to be acquainted with calculus? How can you do this? Well, it is now possible for you to understand this concept even if you are not mathematically inclined.

In this module, we will investigate the manner in which some functions vary, and whether they approach specific values under certain conditions. This analysis will be used in understanding the concept of derivative and definite integral because the definitions of derivative and definite integral depend on the notion of the limit of a function.
2.1 LESSON 1

• An Intuitive Concept of the Limit of a Function
  (Tabular and Graphical Approach)

2.1.1 SPECIFIC OBJECTIVES

At the end of the lesson you should be able to

• determine the behavior of the values of the function \( f(x) \) as \( x \) gets closer and closer to real number \( a \)

• make a table of values for the function \( f \)

• support graphically the conclusion on the limit of the function

2.1.2 PREREQUISITE SKILLS

• Basic algebraic computational and graphing skills

2.1.3 MATERIALS NEEDED

• Graphing paper, calculator

2.1.4 TIME FRAME

• 3 HOURS
2.1.5 LESSON PROPER

You know that a function may be thought of as a set of ordered pairs \((x, y)\) or \((x, f(x))\), with the \(y\)-values related to the \(x\)-values by some rule.

Let a function \(f\) be defined throughout an open interval containing a real number \(a\), except possibly at \(a\) itself. We are often interested in the function value \(f(x)\) when \(x\) is very close to \(a\), but not necessarily equal to \(a\). In fact, in many instances, the number \(a\) is not in the domain of \(f\), i.e. \(f(a)\) is not defined.

The notion of the limit of a function is suggested by the question: “What happens to \(f(x)\) as \(x\) gets nearer and nearer to \(a\) (but \(x \neq a\))? Does \(f(x)\) approach some number \(L\)?” The question implies that we have to find the “limit of \(f(x)\) as \(x\) approaches \(a\), or \(\lim_{x \to a} f(x)\).” We then illustrate this concept in the following examples.

EXAMPLE 1 Investigate the values of the function \(f(x) = x + 2\) as \(x\) approaches 2

We will study the behavior of the linear function \(f(x) = x + 2\) as we choose the \(x\) values in such a way that they are getting closer and closer to 2 from both sides; that is, through the values less than 2 and greater than 2. The table below shows the corresponding values of \(f(x) = x + 2\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>1.5</th>
<th>1.8</th>
<th>1.95</th>
<th>1.999</th>
<th>1.9999</th>
<th>2</th>
<th>2.0001</th>
<th>2.001</th>
<th>2.01</th>
<th>2.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = f(x))</td>
<td>3</td>
<td>3.5</td>
<td>3.8</td>
<td>3.95</td>
<td>3.999</td>
<td>3.9999</td>
<td>4</td>
<td>4.0001</td>
<td>4.001</td>
<td>4.01</td>
<td>4.05</td>
</tr>
</tbody>
</table>
We see from the table that the values of $f(x) = x + 2$ come closer and closer to the value 4 as the values of $x$ get nearer and nearer to 2. In this case, we say that the limit of $f(x)$ as $x$ approaches 2 is 4. Symbolically, we write

$$\lim_{x \to 2}(x + 2) = 4$$

We do not assume that $x = 2$ but rather $x$ gets closer and closer to 2 from either side and this is denoted by $x \to 2$ (x approaches 2). Note that in Figure 1, as $x$ approaches 2 from values less than 2 (denoted as $x \to 2^-$), $x + 2$ is approaching 4. Similarly as $x$ approaches 2 from values greater than 2 (denoted as $x \to 2^+$), $x + 2$ is approaching 4. These results are symbolized as

$$\lim_{x \to 2^+}(x + 2) = 4 \quad \text{and} \quad \lim_{x \to 2^-}(x + 2) = 4.$$

We, therefore, say that $\lim_{x \to 2}(x + 2) = 4$ because $f(x) \to 4$ from both sides.

Observe that when we evaluate $f(x) = x + 2$ at $x = 2$, we have $f(2) = 2 + 2 = 4$ and $\lim_{x \to 2}(x + 2) = 4$.

In some cases, it is not true as we illustrate in the following examples.

**Example 2** Find $\lim_{x \to 1}\left(\frac{x - 1}{x^2 - 1}\right)$
**SOLUTION** Notice that the function \( f(x) = \frac{x-1}{x^2-1} \) is not defined when \( x = 1 \), that is, \( f(x) \) does not exist. If we evaluate the function by substitution,

we have \( \lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1-1}{1^2-1} = \frac{0}{0} \). If the numerator and the denominator both approach zero, we say that the function has an indeterminate form \( \frac{0}{0} \) at \( x = 1 \). But the notion of the limit says that we consider values of \( x \) that are close to \( a \) but not equal to \( a \).

The tables at the right give values of \( f(x) \) for values \( x \) that approach 1 (but not equal to 1). On the basis of the values in the table, we make the guess that

\[
\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}
\]

Example 2 is illustrated by the graph \( f \) in Figure 2. We observe that as \( x \to 1 \) from the left and from the right, \( f(x) \to 0.5 \). Though the graph of the function \( y = \frac{x-1}{x^2-1} \) has a hole at \( x = 1 \), which may or may not appear on your graphics calculator or computer, we have to remember that \( x \to 1 \) implies that \( x \) gets nearer and nearer to 1, but not equal to 1. We are only interested in the small neighborhood of \( x = 1 \), that is, the set of all nearby points lying to the left of \( x = 1 \) and to the right but not in what happens to \( f(x) \) at \( x = 1 \). Therefore, we say that the limit exists even if \( f(x) \) is not defined at \( x = 1 \) (see Figure 2).
Now, let us change \( f \) slightly by giving it the value 2 when \( x = 1 \), and call the resulting function \( g \).

\[
g(x) = \begin{cases} 
\frac{x-1}{x^2-1} & \text{if } x \neq 1 \\
2 & \text{if } x = 1
\end{cases}
\]

What do you think is \( \lim_{x \to 1} g(x) \)?

**EXAMPLE 3** Find \( \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \)

**SOLUTION** The table lists values of the function for several values of \( t \) near 0.

As \( t \) approaches 0, the values of the function seem to approach 0.16666666… and so we guess that

\[
\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{1}{6}
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \frac{\sqrt{t^2 + 9} - 3}{t^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>± 1.0</td>
<td>0.16228</td>
</tr>
<tr>
<td>± 0.5</td>
<td>0.16553</td>
</tr>
<tr>
<td>± 0.1</td>
<td>0.16662</td>
</tr>
<tr>
<td>± 0.05</td>
<td>0.16666</td>
</tr>
<tr>
<td>± 0.01</td>
<td>0.16667</td>
</tr>
</tbody>
</table>

In Example 3, what would have happened if we had taken ever-smaller values of \( t \)? The table at the left shows the results from one calculator; you can see that something strange seems to be happening. If you try these calculations on your own calculator, you might get different values, but eventually you will get the value zero (0) if you make \( t \)
sufficiently small. Does this mean that the answer is really 0 instead of $1/6$? No, the value of the limit is $1/6$, as we will show below. The problem is that the calculator gave false values because $\sqrt{t^2 + 9}$ is very close to 3 when $t$ is too small. (In fact, when $t$ is sufficiently small, a calculator’s value for $\sqrt{t^2 + 9}$ is 3.0000… to as many digits as the calculator is capable of carrying). But if we graph the function, we can see that as $x$ gets closer and closer to zero, $f(x)$ approaches $1/6 \approx 0.166666666667$ (as shown in Figure 4).

Figure 5 shows the graphs of three functions. Note that in parts (b) and (c), $f(a)$ is not defined, $f(a) \neq L$. But in each case, whether $f(x)$ is defined or undefined at $x = a$, still, the $\lim_{x \to a} f(x) = L$.

Figure 5 shows the graphs of three functions. Note that in parts (b) and (c), $f(a)$ is not defined, $f(a) \neq L$. But in each case, whether $f(x)$ is defined or undefined at $x = a$, still, the $\lim_{x \to a} f(x) = L$. 

Figure 5 $\lim_{x \to a} f(x) = L$ in all three cases.
EXAMPLE 4 Let \( f \) be defined by
\[
f(x) = \begin{cases} 
-2 & \text{if } x < 2 \\
2 & \text{if } x > 2
\end{cases}
\]

Find:  
\[ \text{a. } \lim_{x \to 2^-} f(x) \quad \text{b. } \lim_{x \to 2^+} f(x) \quad \text{c. } \lim_{x \to 2} f(x) \]

SOLUTION The graph is shown in Figure 6.

Evaluating the corresponding limits based on the graph in Figure 6, we have
\[
\text{a. } \lim_{x \to 2^-} f(x) = -2 \quad \text{b. } \lim_{x \to 2^+} f(x) = 2
\]
\[ \text{c. } \lim_{x \to 2} f(x) \text{ does not exist}
\]
because \( \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \).

Study Tip

- The \( \lim_{x \to a} f(x) \) does not exist if \( \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \).
Self – Test 2.1  An Intuitive Concept of the Limit of a Function

Graph the function \( y = f(x) \) and determine the \( \lim_{x \to a} f(x) \).

1. \( f(x) = 7, \quad a = 4 \)

2. \( f(x) = -5, \quad a = 4 \)

3. \( f(x) = 2x + 1, \quad a = 3 \)

4. \( f(x) = 7 - 2x, \quad a = -2 \)

5. \( f(x) = \frac{x^2 - 4}{x + 2}, \quad a = -2 \)
Exercise 2.1  An Intuitive Concept of the Limit of a Function

In Exercises 1 through 3, do the following: (a) Use a calculator to tabulate to four decimal places the values of \( f(x) \) for the specified values of \( x \). What does \( f(x) \) appear to be approaching as \( x \) approaches \( c \)? (b) Support you answer in (a) by plotting the graph of \( f \) in a convenient window.

1. \( f(x) = \frac{x^2 - 8x + 16}{x - 4} \); \( c = 4 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>3.0</th>
<th>3.50</th>
<th>3.90</th>
<th>3.99</th>
<th>3.999</th>
<th>5.0</th>
<th>4.5</th>
<th>4.1</th>
<th>4.01</th>
<th>4.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SOLUTION
2. \( f(x) = \frac{x^2 + 4x + 4}{x + 2} \); \( c = -2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-3.0</th>
<th>-2.50</th>
<th>-2.1</th>
<th>-2.01</th>
<th>-2.001</th>
<th>-1.0</th>
<th>-1.5</th>
<th>-1.9</th>
<th>-1.99</th>
<th>-1.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SOLUTION

3. \( f(x) = \frac{3x^2 + 5x - 2}{x + 2} \); \( c = -2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-3.0</th>
<th>-2.50</th>
<th>-2.1</th>
<th>-2.01</th>
<th>-2.001</th>
<th>-1.0</th>
<th>-1.5</th>
<th>-1.9</th>
<th>-1.99</th>
<th>-1.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SOLUTION
2.2 LESSON 2

- An Introduction and Formal Definition of the Limit of a Function
  (Graphical and Analytical Approach)

2.2.1 SPECIFIC OBJECTIVES

At the end of the lesson you should be able to

- define the limit of the function formally
- illustrate the geometric significance of the symbols $\varepsilon$ (epsilon) and $\delta$ (delta)
- support graphically and confirm analytically the conclusion on the limit of the function
- use the definition of the function to prove the limit of a function

2.2.2 PREREQUISITE SKILLS

- Graphing skills, solving and proving inequalities

2.2.3 MATERIALS NEEDED

- Graphing paper, calculator

2.2.4 TIME FRAME

- 3 HOURS
2.2.5 LESSON PROPER

2.2.5.1 An Introduction to the Limit of a Function

Let us now illustrate how to confirm analytically our guess and graphical interpretation about our intuitive concept of the limit of the function. Our results here will pave the way for its formal definition.

Consider the table of values of $f(x) = x + 2$ in Example 1. We see that when $x$ differs from 2 by $\pm 0.001$, (i.e., $x = 1.999$ or $x = 2.001$), $f(x)$ differs from 4 by $\pm 0.001$, (i.e., $f(x) = 3.999$ or $f(x) = 4.001$), and when $x$ differs from 2 by $\pm 0.0001$, $f(x)$ differs from 4 by $\pm 0.0001$.

Let us look at the values of $f(x)$ again. We see that we can make the value of $f(x)$ as close to 4 as we please by taking $x$ close enough to 2; that is, $|f(x) - 4|$ can be made as small as we want by making $|x - 2|$ small enough, but bear in mind that $x$ never takes the value 2.

We will use the Greek letters $\varepsilon$ (epsilon) and $\delta$ (delta) in order to precisely express the above stated condition. Let’s say that for any given positive number $\varepsilon$, there is an appropriately chosen positive number $\delta$ such that, if $|x - 2|$ is less than $\delta$, then $|f(x) - 4|$ will be less than $\varepsilon$. In other words, we say that: Given any positive number $\varepsilon$, we can make $|f(x) - 4| < \varepsilon$ by taking $|x - 2|$ small enough; that is, there is some sufficiently small positive number $\delta$ such that

$$0 < |x - 2| < \delta \quad \text{then} \quad |f(x) - 4| < \varepsilon. \quad (1)$$

This means graphically that if $x$ lies between $2 - \delta$ and $2 + \delta$ on the horizontal axis, then $f(x)$ will lie between $4 - \varepsilon$ and $4 + \varepsilon$ on the vertical axis. This also implies that the $f(x)$
can be restricted to lie between $4 - \varepsilon$ and $4 + \varepsilon$ on the vertical axis by restricting $x$ to lie between $2 - \delta$ and $2 + \delta$ on the horizontal axis (refer to Figure 7).

This shows that for any $\varepsilon > 0$, we can find a $\delta > 0$ such that the statement (1) is true. We can now state that the limit of $f(x)$ as $x$ approaches 2 is equal to 4, or expressed with symbols

$$\lim_{x \to 2} f(x) = 4$$

Let us demonstrate graphically how to choose a suitable $\delta$ for a given $\varepsilon$. Suppose $\varepsilon = 0.1$, that is, we want to restrict $f(x)$ to be between $4 - 0.1$ and $4 + 0.1$, or equivalently between 3.9 and 4.1.

Note that this restriction on $f(x)$ will result to a restriction on $x$ as well. We can determine this restriction on $x$ by solving for $x$ in $f(x) = 3.9$ and $f(x) = 4.1$ from where we get $x = 1.9$ and $x = 2.1$, respectively. From Figures 7 and 9, we can see that if $\varepsilon = 0.1$, then $\delta = 0.1$. Thus, for $\varepsilon = 0.1$, we take a $\delta = 0.1$ and state that

$$0 < |x - 2| < 0.1 \text{ then } |f(x) - 4| < 0.1.$$  

This is statement (1) with $\varepsilon = 0.1$ and $\delta = 0.1$. If you have a graphics calculator or computer with Scientific Notebook or Scientific Workplace, you can get further graphical support by plotting the lines $y = 3.9$ and $y = 4.1$ in the same window as the graph of $f$ (see Figure 9).
If we choose again any smaller positive value for $\varepsilon$, we can find another suitable value for $\delta > 0$. Remember that $\varepsilon$ is chosen arbitrarily and can be as small as desired, and that the value of $\delta$ is dependent on the chosen $\varepsilon$.

**ILLUSTRATION 1  Determining the Value of $\delta$ for Example 1 in Lesson 1**

We shall now determine a $\delta > 0$ for $\varepsilon = 0.001$ such that

\[ 0 < |x - 2| < \delta \quad \text{then} \quad |f(x) - 4| < \varepsilon = 0.001 \]

Refer to our Figure 1 or Figure 7 and observe that the function values increase as $x$ increases. Thus, the figure indicates that we need a value of $x_1$ such that $f(x_1) = 3.999$ and a value of $x_2$ such that $f(x_2) = 4.001$; that is we need an $x_1$ and an $x_2$ such that

\[
\begin{align*}
  f(x_1) &= x_1 + 2 = 3.999 \quad \quad \quad \quad \quad x_2 + 2 = 4.001 \\
  x_1 &= 1.999 \quad \quad \quad \quad \quad \quad \quad \quad \quad x_2 = 2.001
\end{align*}
\]

Because $2 - 1.999 = 0.001$ and $2.001 - 2 = 0.001$, we choose a $\delta = 0.001$ so that we have the statement

\[ 0 < |x - 2| < 0.001 \quad \text{then} \quad |f(x) - 4| < 0.001 \]

Confirming Analytically the Choice of $\delta$

By using the properties of inequalities, we shall now confirm analytically our choice of $\delta$. In our solution, we will use the symbols $\Rightarrow$ and $\Leftrightarrow$. The arrow $\Rightarrow$ means *implies* and the double arrow $\Leftrightarrow$ means *double implication*, in other words, the statement preceding it and the statement following it are equivalent.

We wish to determine a $\delta > 0$ such that

\[ 0 < |x - 2| < \delta \quad \text{then} \quad |f(x) - 4| < 0.001 \]
\[ \Leftrightarrow \text{ if } 0 < |x - 2| < \delta \text{ then } |(x + 2) - 4| < 0.001 \]
\[ \Leftrightarrow \text{ if } 0 < |x - 2| < \delta \text{ then } |x - 2| < 0.001 \]

This statement indicates that a suitable choice for \(\delta\) is 0.001. We now have the following arguments:

\[
0 < |x - 2| < 0.001 \\
\implies 0 < |(x+2) - 4| < 0.001 \\
\implies 0 < |f(x) - 4| < 0.001
\]

We have confirmed analytically that

\[
\text{if } 0 < |x - 2| < 0.001 \text{ then } |f(x) - 4| < 0.001 \quad (2)
\]

In our solutions, any positive number less than 0.001 can be used in place of 0.001 as required \(\delta\). Observe this fact in our table of values and in Figures 1 & 7. Furthermore, if \(0 < |x - 2| < \lambda\) then \(|f(x) - 4| < 0.001\) because any number \(x\) satisfying the inequality \(0 < |x - 2| < \lambda\) also satisfies the inequality \(0 < |x - 2| < 0.001\). Thus, from our solutions for determining a \(\delta > 0\) and confirming a choice of \(\delta\) for a specific \(\epsilon\), we will learn that if for any \(\epsilon > 0\) we can find a \(\delta > 0\) such that

\[
\text{if } 0 < |x - 2| < \delta \text{ then } |f(x) - 4| < \epsilon
\]

We shall have established that \(\lim_{x \to 2} (x + 2) = 4\).

**ILLUSTRATION 2  Determining the Value of \(\delta\) for Example 2 in Lesson 1**

Let us consider now Example 2 and the graph is sketched in Figure 2. The graph of \(f\) is not defined at \(x = 1\). Since we are interested at the values of \(f(x)\) with \(x \neq 1\), we can...
apply the same argument above and conclude that for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
\text{if } 0 < |x - 1| < \delta \quad \text{then} \quad |f(x) - \frac{1}{2}| < \varepsilon \quad \text{so that} \quad \lim_{x \to 1} f(x) = \frac{1}{2}
\]

We shall now determine a \( \delta > 0 \) for \( \varepsilon = 0.01 \) and confirm analytically our choice of \( \delta \) such that if \( 0 < |x - 1| < \delta \) then \( |f(x) - \frac{1}{2}| < 0.01 \).

Figure 2 shows a piece of the graph of \( f \) in the neighborhood of the point \((1, \frac{1}{2})\). If \( x > 0 \), the function values decrease, but they only approach zero, as the values of \( x \) increase. This indicates that we need a positive value of \( x_1 \) such that \( f(x_1) = 0.51 \) and a positive value of \( x_2 \) such that \( f(x_2) = 0.49 \); that is, we need an \( x_1 > 0 \) and an \( x_2 > 0 \), but \( x \neq 1 \), such that

\[
\frac{x_1 - 1}{x_1^2 - 1} = 0.51 \quad \frac{x_2 - 1}{x_2^2 - 1} = 0.49
\]

\[
0.51x_1^2 - 0.51 = x_1 - 1 \quad 0.49x_2^2 - 0.49 = x_2 - 1
\]

\[
0.51x_1^2 - x_1 + 0.49 = 0 \quad 0.49x_2^2 - x_2 + 0.51 = 0
\]

\[
x_1 \approx 0.9608 \text{ or } 1 \quad x_2 \approx 1.0408 \text{ or } 1
\]

Then, \( 1 - 0.9608 = 0.0392 \) and \( 1.0408 - 1 = 0.0408 \). Since we want a smaller \( \delta \) and \( 0.0392 < 0.0408 \), we then choose \( \delta = 0.0392 \) so that we have the statement

\[
\text{if } 0 < |x - 1| < 0.0392 \quad \text{then} \quad |f(x) - \frac{1}{2}| < 0.01
\]

Note that any positive number less than 0.0392 can be selected as the required \( \delta \).
Confirming the Choice of $\delta$ for Example 2

We shall now confirm analytically our choice of $\delta$ in the preceding solution using the properties of inequalities.

**SOLUTION**  We want to find a $\delta > 0$ such that

\[
\text{if } 0 < |x - 1| < \delta \quad \text{then} \quad |f(x) - 1/2| < 0.01
\]

\[
\iff \text{if } 0 < |x - 1| < \delta \quad \text{then} \quad \left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| < 0.01
\]

\[
\iff \text{if } 0 < |x - 1| < \delta \quad \text{then} \quad \left|\frac{2x - 2 - x^2 + 1}{2(x^2-1)}\right| < 0.01
\]

\[
\iff \text{if } 0 < |x - 1| < \delta \quad \text{then} \quad \left|\frac{-(x^2 - 2x + 1)}{2(x^2-1)}\right| < 0.01
\]

\[
\iff \text{if } 0 < |x - 1| < \delta \quad \text{then} \quad \left|\frac{-(x-1)(x-1)}{2(x-1)(x+1)}\right| < 0.01
\]

\[
\iff \text{if } 0 < |x - 1| < \delta \quad \text{then} \quad \left|\frac{-(x-1)}{2(x+1)}\right| < 0.01
\]

\[
\iff \text{if } 0 < |x - 1| < \delta \quad \text{then} \quad \frac{1}{2} \left|\frac{1}{x+1}\right| |x-1| < 0.01
\]

Notice on the right-hand side of the statement that in addition to the factor $|x-1|$, we have the factors $\frac{1}{2}$ and $\left|\frac{1}{x+1}\right|$. We need to obtain, therefore, an inequality involving $\frac{1}{2} \left|\frac{1}{x+1}\right|$. We do this by putting a restriction on the $\delta$ we are seeking. Let us restrict our $\delta$ to
be less than or equal to 0.05, which seems reasonable because we want a \( \delta \) small enough.

Then

\[
0 < |x - 1| < \delta \quad \text{and} \quad \delta \leq 0.05
\]

\[
\Rightarrow 0 < |x - 1| < 0.05
\]

\[
\Rightarrow -0.05 < x - 1 < 0.05
\]

\[
\Rightarrow 0.95 < x < 1.05
\]

\[
\Rightarrow 1.95 < x < 2.05
\]

\[
\Rightarrow \frac{1}{2.05} < \frac{1}{x+1} < \frac{1}{1.95}
\]

\[
\Rightarrow \left| \frac{1}{x+1} \right| < \frac{1}{1.95}
\]

\[
\Rightarrow \frac{1}{2} \left| \frac{1}{x+1} \right| < \frac{1}{2 \times 1.95} = \frac{1}{3.9}
\]

Thus,

\[
0 < |x - 1| < \delta \quad \text{and} \quad \delta \leq 0.05
\]

\[
\Rightarrow 0 < |x - 1| < \delta \quad \text{and} \quad \frac{1}{2} \left| \frac{1}{x+1} \right| < \frac{1}{3.9}
\]

\[
\Rightarrow \frac{1}{2} \left| \frac{1}{x+1} \right| |x - 1| < \delta \left( \frac{1}{3.9} \right)
\]

Remember that our goal is statement (3). We should require \( \delta \left( \frac{1}{3.9} \right) \leq 0.01 \Leftrightarrow \delta \leq 0.039. \)

We have now put two restrictions on \( \delta \): \( \delta \leq 0.05 \) and \( \delta \leq 0.039. \) So that both restrictions hold, we take \( \delta \leq 0.039, \) the smaller of the two numbers. Using this \( \delta, \) we have the following argument:
\[ 0 < |x - 1| < 0.039 \]

\[ \Rightarrow |x - 1| < 0.039 \quad \text{and} \quad \left| \frac{1}{x+1} \right| < \frac{1}{1.95} \]

\[ \Rightarrow |x - 1| < 0.039 \quad \text{and} \quad \frac{1}{2} \left| \frac{1}{x+1} \right| < \frac{1}{3.9} \]

\[ \Rightarrow \frac{1}{2} \left| \frac{1}{x+1} \right| |x - 1| < (0.039) \left( \frac{1}{3.9} \right) \]

\[ \Rightarrow \left| \frac{1}{2} \left( \frac{x - 1}{x + 1} \right) \right| < 0.01 \quad \Rightarrow \quad \left| \frac{(1-x)(x-1)}{2(x-1)(x+1)} \right| < 0.01 \]

\[ \Rightarrow \left| \frac{-x^2 + 2x - 1}{2(x^2 - 1)} \right| < 0.01 \]

\[ \Rightarrow \left| \frac{2x - 2 - x^2 + 1}{2(x^2 - 1)} \right| < 0.01 \]

\[ \Rightarrow \left| \frac{2(x-1) - (x^2 - 1)}{2(x^2 - 1)} \right| < 0.01 \]

\[ \Rightarrow \left| \frac{x - 1}{x^2 - 1} - \frac{1}{2} \right| < 0.01 \]

We have therefore determined \( \delta \) so that statement (3) holds true. Since 0.039 < 0.0392 and it conforms with the restrictions on \( \delta \), we have confirmed our choice of \( \delta \) for Example 2.
2.2.5.2 LIMIT: THE FORMAL DEFINITION

The limit of a function \( f(x) \) as \( x \) approaches \( a \) is \( L \) denoted by \( \lim_{x \to a} f(x) = L \) if and only if for any positive number \( \varepsilon \) (no matter how small) there exists a positive number \( \delta \) (dependent on \( \varepsilon \)) such that
\[
|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta \quad (\text{see Figure 10}).
\]

This means that if the statement \( \lim_{x \to a} f(x) = L \) is true, then we can find \( \delta \) such that \( f(x) \) can be made as close to \( L \) as we please (i.e. within \( \varepsilon \) of \( L \)) by substituting any value of \( x \) within a distance \( \delta \) of \( a \) into \( f(x) \).

**EXAMPLE 1** By applying the definition of the limit, prove that
\[
\lim_{x \to 2} (5x - 4) = 6
\]

**SOLUTION** Since \( 5x - 4 \) is defined for all real numbers, any open interval containing 2, except possibly at 2, is also defined. Now, we must show that for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that
\[
|5(x - 2) - 6| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta \quad (1)
\]

\[
\Leftrightarrow \quad |5(x - 2)| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta
\]

\[
\Leftrightarrow \quad 5|x - 2| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta
\]

\[
\Leftrightarrow \quad |x - 2| < \frac{\varepsilon}{5} \quad \text{whenever} \quad 0 < |x - 2| < \delta
\]
This statement indicates that $\varepsilon/5$ is a satisfactory $\delta$. With this choice of $\delta$, we have the following argument:

$$0 < |x - 2| < \delta$$

$$\Rightarrow 5|x - 2| < 5\delta$$

$$\Rightarrow |5x - 10| < 5\delta$$

$$\Rightarrow |(5x - 4) - 6| < \varepsilon \quad \text{(since } \delta = \frac{1}{5} \varepsilon \text{)}$$

We have, therefore, established that if $\delta = \frac{1}{5} \varepsilon$, statement (1) holds true. This proves that

$$\lim_{x \to 2} (5x - 4) = 6 \quad \text{(see Figure 11)}.$$  

In particular, if $\varepsilon = 0.1$, then we take $\delta = \frac{1}{5} (0.1)$, that is, 0.02.

Any number less than $\varepsilon/5$ can also be used as a required $\delta$.

**EXAMPLE 2** Prove that $\lim_{x \to 4} x^2 - 3x - 10 = -6$

**SOLUTION** Given any $\varepsilon > 0$, we want to find $\delta > 0$ such that

if $0 < |x - 4| < \delta$ then $|(x^2 - 3x - 10) + 6| < \varepsilon \quad (2)$

$\Leftrightarrow$ if $0 < |x - 4| < \delta$ then $|x^2 - 3x - 4| < \varepsilon$

$\Leftrightarrow$ if $0 < |x - 4| < \delta$ then $|(x + 1)(x - 4)| < \varepsilon$

$\Leftrightarrow$ if $0 < |x - 4| < \delta$ then $|x + 1||x - 4| < \varepsilon$
Because we have a factor $|x+1|$ aside from the factor $|x-4|$, we need to obtain, therefore, an inequality $|x+1|$ by putting a restriction on $\delta$ we are seeking.

Let $\delta \leq \frac{1}{10}$, then

\[
0 < |x-4| < \delta = \frac{1}{10}
\]

\[
\Rightarrow -\frac{1}{10} < x - 4 < \frac{1}{10}
\]

\[
\Rightarrow \frac{39}{10} < x < \frac{41}{10}
\]

\[
\Rightarrow \frac{49}{10} < x + 1 < \frac{51}{10}
\]

\[
\Rightarrow |x+1| < \frac{51}{10}
\]

Thus $0 < |x-4| < \delta$ and $\delta \leq \frac{1}{10}$

\[
\Rightarrow 0 < |x-4| < \delta \text{ and } |x+1| < \frac{51}{10}
\]

\[
\Rightarrow |x-4||x+1| < \frac{51}{10}\delta
\]

\[
\Rightarrow \delta \leq \frac{10}{51}\varepsilon
\]
Since our goal is statement (2), we need to choose from $\delta \{ \frac{10}{51} \varepsilon, \frac{1}{10} \}$, the smaller of the two numbers. Using $\delta = \frac{10}{51} \varepsilon$, we have the following argument:

$$0 < |x - 4| < \delta = \frac{10}{51} \varepsilon$$

$$\Rightarrow \ |x - 4| < \frac{10}{51} \varepsilon .$$

Since $|x + 1| < \frac{51}{10}$, we have

$$|x - 4| |x + 1| < \frac{51 \cdot 10}{10 \cdot 51} \varepsilon$$

$$\Rightarrow \ |(x + 1)(x - 4)| < \varepsilon$$

$$\Rightarrow \ |x^2 - 3x - 10 + 6| < \varepsilon$$

We have now proven statement (2) $\blacksquare$
Self–Test 2.2 Introduction & Formal Definition of the Limit of a Function

A. For Exercises 1 & 2, use the arguments similar to those in the illustrations to determine a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

1. $f(x) = 3x + 7; \ a = 2; \ L = 13; \ \varepsilon = 0.5$

2. $f(x) = 2x^2 + 5x + 3; \ a = -3; \ L = 6; \ \varepsilon = 0.1$

3. Confirm analytically, by using the properties of inequalities, your choice of $\delta$ in Exercises 1 & 2.

B. In Exercises 1 – 2, do the following: (a) Use a calculator to tabulate to four decimal places the values of $f(x)$ for the specified values of $x$. What does $f(x)$ appear to be approaching as $x$ approaches $c$? (b) Support your answer in (a) by plotting the graph of $f$ in a convenient window. (c) Prove your answer in (a) by using the formal definition of a limit.

1. $f(x) = \frac{2x^2 + 3x - 2}{x^2 - 6x - 16}; \ c = -2$

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
x & -3.0 & -2.50 & -2.10 & -2.01 & -2.001 & -1.0 & -1.5 & -1.9 & -1.99 & -1.999 \\
\hline
f(x) & & & & & & & & & & \\
\hline
\end{array}
\]

2. $f(x) = \frac{x^2 + 5x + 6}{x^2 - x - 12}; \ c = -3$

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
x & -4.0 & -3.50 & -3.10 & -3.01 & -3.001 & -2.0 & -2.5 & -2.9 & -2.99 & -2.999 \\
\hline
f(x) & & & & & & & & & & \\
\hline
\end{array}
\]
Exercise 2.2  Introduction & Formal Definition of the Limit of a Function

Prove that the limit is the indicated number by applying its formal definition

1. \( \lim_{x \to 0} 1 = 1 \)

2. \( \lim_{x \to 0} -15 = -15 \)

3. \( \lim_{x \to 4} \sqrt{x} = 2 \)

4. \( \lim_{x \to 1} (1 + 3x) = 4 \)

5. \( \lim_{x \to -2} \frac{1}{x} = -\frac{1}{2} \)

6. \( \lim_{x \to -2} (7 - 2x) = 11 \)

7. \( \lim_{x \to 3} (x^2 + 3x + 2) = 20 \)

8. \( \lim_{x \to -2} \frac{x + 2}{x^2 - 4} = -\frac{1}{4} \)

9. \( \lim_{x \to 1} \frac{x - 1}{1 - \sqrt{x}} = -2 \)

10. \( \lim_{x \to 1} \frac{x - 1}{\sqrt{x} - \sqrt{2} - x} = 1 \)
2.3 LESSON 3 Calculating Limits Using the Limit Laws

2.3.1 SPECIFIC OBJECTIVES

At the end of the lesson you should be able to

- enumerate the different properties of limits, called the limit laws
- use these properties of limits to evaluate limits

2.3.2 PREREQUISITE SKILLS

- Rationalizing the numerator or denominator and Factoring

2.3.3 LESSON PROPER

In Lesson 1, we used calculators and graphs to determine the values of limits. We also supported graphically and confirmed analytically our conclusion on the limit of the function. Lately, we defined the limit of a function and used the definition to prove the limit. In this lesson, we use the following properties of limits, called the Limit Laws, to easily calculate limits.

These five laws can be stated verbally as follows:

**Sum Law**  The limit of a sum is the sum of the limit.

**Difference Law**  The limit of a difference is the difference of the limits.

**Constant Multiple Law**  The limit of a constant times a function is the constant times the limit of the function.

**Product Law**  The limit of a product is the product of the limits.

**Quotient Law**  The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not zero).

Suppose that $c$ is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$  if  $\lim_{x \to a} g(x) \neq 0$
EXAMPLE 1 Evaluate the following limits and justify each step
a. \( \lim_{x \to 5} (2x^2 - 3x + 4) \)  
   b. \( \lim_{x \to 2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \)

SOLUTION

a. \( \lim_{x \to 5} (2x^2 - 3x + 4) = \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} (4) \) (by Laws 2 & 1)
   \[ = 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 \] (by 3)
   \[ = 2(5^2) - 3(5) + 4 \] (by 9, 8 & 7)
   \[ \lim_{x \to 5} (2x^2 - 3x + 4) = 39 \]

b. \( \lim_{x \to 2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \)
   \[ = \frac{\lim_{x \to 2} (x^3 + 2x^2 - 1)}{\lim_{x \to 2} (5 - 3x)} \] (by Law 5)
   \[ = \frac{\lim_{x \to 2} x^3 + 2 \lim_{x \to 2} x^2 - \lim_{x \to 2} 1}{\lim_{x \to 2} 5 - 3 \lim_{x \to 2} x} \] (by 1, 2, & 3)
   \[ \lim_{x \to 2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \]

Observe that we can evaluate limits by substitution. This can only be done when the limits of the numerator and the denominator are not equal to zero.

Limit Laws

If we use the Product Law repeatedly, we obtain the following Power Laws and Root Laws.

6. \( \lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \)
   where \( n \) is a positive integer

7. \( \lim_{x \to a} x^n = a^n \) where \( n \) is a positive integer

8. \( \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \) where \( n \) is a positive integer
   (If \( n \) is even, we assume that \( a \geq 0 \).)

9. \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \)
   (If \( n \) is even, we assume that \( \lim_{x \to a} f(x) > 0 \))
EXAMPLE 2  Find \( \lim_{x \to 3} \frac{\sqrt{x+1}}{x} \)

SOLUTION  Using the Law 9 and substituting 3 to \( x \), we have

\[
\lim_{x \to 3} \frac{\sqrt{x+1}}{x} = \lim_{x \to 3} \frac{\sqrt{x+1}}{\lim_{x \to 3} x} = \frac{\sqrt{3+1}}{3} = \frac{2}{3}
\]

EXAMPLE 3  Evaluate the limit of the following.

a. \( \lim_{x \to 0} \frac{x}{\sqrt{x}} \)  

b. \( \lim_{x \to -3} \frac{x^3 + 27}{x + 3} \)  

c. \( \lim_{x \to 9} \frac{\sqrt{x - 3}}{x - 9} \)  

d. \( \lim_{x \to 1} \frac{x - 1}{1 - \sqrt{x}} \)

SOLUTIONS  For problems a, c and d, we have to rationalize first the numerator or the denominator. For problem b, we need to factor the numerator. Then, we evaluate the limit.

a. \( \lim_{x \to 0} \frac{x}{\sqrt{x}} = \lim_{x \to 0} x \cdot \frac{\sqrt{x}}{\sqrt{x}} \\
= \lim_{x \to 0} \frac{x \sqrt{x}}{x} \). Since \( x \neq 0 \), we can cancel out \( x \). Thus,

\( \lim_{x \to 0} \sqrt{x} = \sqrt{0} = 0 \)

b. \( \lim_{x \to -3} \frac{x^3 + 27}{x + 3} = = \\
\lim_{x \to -3} \frac{(x+3)(x^2 - 3x + 9)}{x + 3} \).

Since \( x \neq -3, \ x + 3 \neq 0 \). Thus, we have

\( \lim_{x \to -3} (x^2 - 3x + 9) = (-3)^2 - 3(-3) + 9 = 27 \)
c. \[ \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \lim_{x \to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}. \]

Since \( x \neq 9 \), \( x - 9 \neq 0 \). We get

\[ \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6} \]

d. \[ \lim_{x \to 1} \frac{x - 1}{1 - \sqrt{x}} = \lim_{x \to 1} \frac{x - 1}{1 - \sqrt{x}} \cdot \frac{1 + \sqrt{x}}{1 + \sqrt{x}} = \lim_{x \to 1} \frac{(x - 1)(1 + \sqrt{x})}{1 - x} = \lim_{x \to 1} \frac{(x - 1)(1 + \sqrt{x})}{-(x - 1)} \]

\[ = \lim_{x \to 1} (1 + \sqrt{x}) = -(1 + \sqrt{1}) = -2 \]

**EXAMPLE 4** Given

\[ f(x) = \frac{\sqrt{x} - 2}{x - 4}, \text{ evaluate } \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \]

**SOLUTION** The Limit Laws cannot be applied to the quotient \( \frac{\sqrt{x} - 2}{x - 4} \) because \( \lim_{x \to 4} x - 4 = 0 \). To simplify the quotient, we rationalize first the numerator by multiplying the numerator and denominator by \( \sqrt{x} + 2 \). The solution is as follows:

\[ \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{(x - 4)}{(x - 4)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} \]

\[ = \frac{\lim_{x \to 4} 1}{\lim_{x \to 4} (\sqrt{x} + 2)} \]  

(Special Limit (a) & Law 1)

\[ = \frac{1}{\lim_{x \to 4} \sqrt{x} + \lim_{x \to 4} 2} \]

(Special Limit (a) & Law 9)

\[ = \frac{1}{\sqrt{4} + 2} \]

\[ = \frac{1}{4} \]
Rationalizing the Denominator or Numerator

In working with the limit of a quotient involving radicals, it is often convenient to move the radical expression from the denominator to the numerator, or vice versa.

- If the numerator or denominator is a monomial containing \( \sqrt{x} \), multiply the expression by \( \frac{\sqrt{x}}{\sqrt{x}} \). Since \( \sqrt{x} \cdot \sqrt{x} = \sqrt{x^2} = x \).

- If the numerator or denominator is a binomial containing \( \sqrt{x} + \sqrt{y} \), multiply the expression by \( \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y}} \). Since, \((x + y)(x - y) = x^2 - y^2 \).

- If the numerator or denominator is a binomial containing \( \sqrt{x} - \sqrt{y} \), multiply the expression by \( \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \). Since, \((x - y)(x + y) = x^2 - y^2 \).

- If the numerator or denominator is a binomial containing \( \sqrt[3]{x} + \sqrt[3]{y} \), multiply the expression by \( \frac{\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2}}{\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2}} \). Since, \((x + y)(x^2 - xy + y^2) = x^3 + y^3 \).

- If the numerator or denominator is a binomial containing \( \sqrt[3]{x} - \sqrt[3]{y} \), multiply the expression by \( \frac{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} \). Since, \((x - y)(x^2 + xy + y^2) = x^3 + y^3 \).

- If the numerator or denominator is a trinomial containing \( \sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2} \), multiply the expression by \( \frac{\sqrt[3]{x} + \sqrt[3]{y}}{\sqrt[3]{x} + \sqrt[3]{y}} \). Since, \((x^2 - xy + y^2)(x + y) = x^3 + y^3 \).

- If the numerator or denominator is a trinomial containing \( \sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2} \), multiply the expression by \( \frac{\sqrt[3]{x} - \sqrt[3]{y}}{\sqrt[3]{x} - \sqrt[3]{y}} \). Since, \((x^2 + xy + y^2)(x - y) = x^3 - y^3 \).
Self-Test 2.3.A  CALCULATING LIMIT USING LIMIT LAWS

In items 1-10, find the limit and, when appropriate, indicate the limit laws being applied.

1. \( \lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1} \)

3. \( \lim_{x \to 2} \frac{3x^2 + 5x - 2}{x + 2} \)

5. \( \lim_{x \to 2} \frac{x^2 - 4}{x^2 - 8x + 12} \)

7. \( \lim_{x \to 7} \frac{x^2 + 14x + 49}{x - 3} \)

9. \( \lim_{x \to 2} \frac{x^4 - 16}{4 - x^2} \)

2. \( \lim_{x \to 3} \frac{x + 3}{x^2 - x - 15} \)

4. \( \lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3} \)

6. \( \lim_{x \to \frac{1}{4}} \frac{12x^2 - x - 1}{4x + 1} \)

8. \( \lim_{x \to 0} \left( x \left( \frac{1}{1 - \frac{1}{x}} \right) \right) \)

10. \( \lim_{x \to 1} \frac{\sqrt{x^4 - 1}}{\sqrt{x - 1}} \)
11. Charle’s law of gases states that if the pressure remains constant, then the relationship between the volume \( V \) that a gas occupies and its temperature \( T \) (in °C) is given by

\[
V = V_0 \left(1 + \frac{1}{273}T\right)
\]

where the temperature \( T = -273°\text{C} \) is absolute zero. Evaluate \( \lim_{T \to -273} V \).

12. A drove of 100 goats is relocated to a small island in Aurora Province. The drove increases rapidly, but eventually the food resources of the island dwindle and the population declines. Suppose that the number \( N(t) \) of goats present after \( t \) years is given by

\[
N(t) = -t^2 + 21t + 100.
\]

Evaluate \( \lim_{t \to 10} N(t) \).

13. A population of flies is growing in a large container. The number of flies \( P \) (in hundreds) after \( t \) weeks is given by

\[
P(t) = -t^4 + 15t^2 + 5.
\]

Evaluate \( \lim_{t \to 9} P \).

**Self-Test 2.3.B  LIMIT LAWS**

Evaluate the limit, if it exists. Indicate the limit laws being applied, when appropriate.

1. \( \lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1} \)
2. \( \lim_{x \to 2} \frac{x - 2}{x^2 - 4x - 4} \)
3. \( \lim_{x \to 2} \frac{x^2 - 6x + 8}{x - 2} \)
4. \( \lim_{x \to 4} \frac{\sqrt{x} - 4}{x - 4} \)
5. \( \lim_{x \to 1} \frac{x - 1}{1 - \sqrt{x}} \)
6. \( \lim_{x \to 9} \frac{2x - 18}{2\sqrt{x} - 6} \)
7. \( \lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1} \)
8. \( \lim_{x \to 0} \frac{(x - 1)^3 - 1}{x} \)
9. \( \lim_{x \to -1} \frac{1 + \sqrt[3]{x}}{x + 1} \)
10. \( \lim_{x \to 8} \frac{x - 8}{\sqrt[3]{x} - 2} \)
11. \[ \lim_{x \to 3} \frac{(x-1)^5}{x^5 - 1} \]

12. \[ \lim_{x \to 2} \frac{2x - 4}{x^3 - 2x^2} \]

13. \[ \lim_{x \to 1} \frac{x^4 + 4x^3 + 3x^2 - 4x - 4}{x + 1} \]

14. The temperature \( T \) (in °C) at which water boils may be approximated by the formula
   \[ T = 100.862 - 0.0415 \sqrt{h + 431.03} \]

where \( h \) is the elevation (in meters above sea level). Evaluate \( \lim_{h \to 4,000} T \).

15. According to Einstein’s Theory of Relativity, the length of an object depends on its velocity \( v \). Einstein also proved that the mass \( m \) of an object is related to \( v \) by the formula
   \[ m = m_0 \sqrt{1 - \frac{v^2}{c^2}} \]

where \( m_0 \) is the mass of the object at rest. Investigate \( \lim_{v \to c} m \) and use the result to justify that \( c \) is the ultimate speed in the universe.

Exercise 2.3  LIMIT LAWS

Student: ____________________________  Score: ____________________________

Evaluate the limit.

1. \[ \lim_{x \to 3} (x^3 + 4x^2 - 6x - 5) \]

2. \[ \lim_{x \to \frac{1}{2}} \frac{x^2 - x - 2}{2x + 5} \]

SOLUTION  SOLUTION
3. \( \lim_{y \to 0} \frac{(3 + y)^3 - 27}{y} \)

4. \( \lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x^2 - 2x} \)

5. \( \lim_{x \to 0} \frac{\sqrt{x} + 2 - \sqrt{2}}{x} \)
2.4 LESSON 4    ONE-SIDED LIMITS

♦ Right – Hand Limits
♦ Left – Hand Limits
♦ Two – Sided Limits

2.4.1 SPECIFIC OBJECTIVES

At the end of the lesson you should be able to

- evaluate the limit of a function f(x) as x approaches a certain value from either side
- sketch the graph and examine the behavior of the function f(x) as x approaches a certain value

2.4.2 PREREQUISITE SKILLS

- Algebraic computational and graphing skills

2.4.3 TIME FRAME

- 4 HOURS

2.4.4 LESSON PROPER

We discussed in Lesson 1 that f(x) may approach two values as x approaches a certain number from the left and from the right (see Figure 5). In such a case, the limit is not defined or does not exist but the right – hand and left – hand limits exist.

In this lesson, we will discuss the behavior of the function as x approaches from either side to a certain value.
The Right – Hand Limit

The right – hand limit of a function \( f(x) \) at a point \( a \) is the limit of the function at \( a \) as \( x \) approaches \( a \) from decreasing values of \( a \), (i.e. as \( x \) approaches \( a \) from the right). We write the right – hand limit by the notation

\[
\lim_{x \to a^-} f(x) = L \quad \text{or} \quad f(x) = L \quad \text{as} \quad x \to a^-
\]

The Left – Hand Limit

The left – hand limit of a function \( f(x) \) at a point \( a \) is the limit of the function at \( a \) as \( x \) approaches \( a \) from increasing values of \( a \), (i.e. as \( x \) approaches \( a \) from the left). We write the left – hand limit by the notation

\[
\lim_{x \to a^+} f(x) = L \quad \text{or} \quad f(x) = L \quad \text{as} \quad x \to a^+
\]

The Two – Sided Limit

The limit exists if and only if the right – hand and the left – hand limits both exist and are equal to \( L \). Thus, we say that the function has two – sided limit.

\[
\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L
\]

Example 1 Given a signum function

\[
\text{sgn } x = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]

Evaluate \( \lim_{x \to 0^-} \text{sgn } x \), \( \lim_{x \to 0^+} \text{sgn } x \), and \( \lim_{x \to 0} \text{sgn } x \)

SOLUTION The graph is shown in Figure 13.

Evaluating the limit, yields

\[
\lim_{x \to 0^-} \text{sgn } x = -1 \quad \text{and} \quad \lim_{x \to 0^+} \text{sgn } x = 1
\]
The limit does not exist since the left hand and the right hand limits are not equal. Thus, 
\[ \lim_{x \to 0} \text{sgn } x \] 
does not exist.

Example 2 The Greatest Integer Function is defined by \( \lfloor x \rfloor = \) the largest integer that is less than or equal to \( x \). (For instance, \( \lfloor 4 \rfloor = 4 \), \( \lfloor 4.8 \rfloor = 4 \), \( \lfloor \pi \rfloor = 3 \), \( \lfloor \sqrt{2} \rfloor = 1 \)). Show that 
\[ \lim_{x \to 3} \lfloor x \rfloor \] 
does not exist.

SOLUTION The graph of the greatest integer function is shown in Figure 14. Since \( \lfloor x \rfloor = 3 \) for \( 3 \leq x < 4 \), we have 
\[ \lim_{x \to 3^+} \lfloor x \rfloor = \lim_{x \to 3^+} 3 = 3. \] Since \( \lfloor x \rfloor = 2 \) for \( 2 \leq x < 3 \), we have 
\[ \lim_{x \to 3^-} \lfloor x \rfloor = \lim_{x \to 3^-} 2 = 2. \] Because these one-sided limits are not equal, 
\[ \lim_{x \to 3} \lfloor x \rfloor \] 
does not exist.

Example 3 Evaluate the limit of \( f(x) = \sqrt{x} \) as \( x \) approaches 0 from the right.

SOLUTION The graph of \( f(x) = \sqrt{x} \) is sketched in Figure 15. Notice that the function is defined only for \( x \geq 0 \). Now, evaluating the limit of the function as \( x \)
approaches 0 from the right, we get \( \lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0 \).

**Example 4** Let \( f \) be defined by

\[
f(x) = \begin{cases} 
2 & \text{if } x = 2 \\
-4 & \text{if } x < 2 
\end{cases}
\]

(a) Sketch the graph of \( f \)

(b) Find \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \)

**SOLUTION**

(a). The graph of \( f \) is sketched in Figure 16.

(b). We use \( f(x) = -4 \) to evaluate the limit as \( x \) approaches 2 from the left. Hence,

\[
\lim_{x \to 2^-} f(x) = -4
\]

Study Tip

- The \( \lim_{x \to a} f(x) = L \) if and only if
  \[
  \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L
  \]

Notice that \( f(x) = 2 \) for \( x = 2 \). Now, evaluating the limit of \( f(x) \) as \( x \) approaches 2 from the right, we have

\[
\lim_{x \to 2^+} f(x) \text{ does not exist}
\]

Now, think of this function \( g \) defined by

\[
g(x) = \begin{cases} 
|x| & \text{if } x \neq 0 \\
3 & \text{if } x = 0 
\end{cases}
\]

a) How does the graph of \( g \) look like?

b) What do you think is \( \lim_{x \to 0} g(x) \)?
Self - Test 2.4   ONE – SIDED LIMITS

In problems 1 – 5, evaluate the limit as \( x \) approaches \( a \) from its either side.

1. \( y = 3x + 2, \ a = -1 \)  
2. \( f(x) = x^2 + x + 4, \ a = 0 \)  
3. \( y = \frac{x^2 + 2x +1}{x+1}, \ a = -1 \)  
4. \( f(x) = \frac{x^2 - 1}{x+1}, \ a = -1 \)  
5. \( f(x) = x\sqrt{x+1}, \ a = 3 \)

In exercises 6 through 10, sketch the graph of the function and find the indicated limit if it exists; if the limit does not exist, state the reason.

6. \( f(x) = \begin{cases} 
2 & \text{if } x < 1 \\
-1 & \text{if } x = 1 \\
-3 & \text{if } x > 1 
\end{cases} \)  
(a) \( \lim_{x \to 1^+} f(x) \)  
(b) \( \lim_{x \to 1^-} f(x) \)  
(c) \( \lim_{x \to 1} f(x) \)

7. \( f(t) = \begin{cases} 
 t + 4 & \text{if } t \leq -4 \\
 4 - t & \text{if } t > -4 
\end{cases} \)  
(a) \( \lim_{t \to -4^+} f(t) \)  
(b) \( \lim_{t \to -4^-} f(t) \)  
(c) \( \lim_{t \to -4} f(t) \)

8. \( f(x) = \begin{cases} 
 x^2 & \text{if } x \leq 2 \\
 8 - 2x & \text{if } x > 2 
\end{cases} \)  
(a) \( \lim_{x \to 2^+} f(x) \)  
(b) \( \lim_{x \to 2^-} f(x) \)  
(c) \( \lim_{x \to 2} f(x) \)

9. \( f(x) = |x - 5| \)  
(a) \( \lim_{x \to 5^+} f(x) \)  
(b) \( \lim_{x \to 5^-} f(x) \)  
(c) \( \lim_{x \to 5} f(x) \)

10. \( f(x) = \begin{cases} 
 x^2 - 4 & \text{if } x < 2 \\
 4 & \text{if } x = 2 \\
 4 - x^2 & \text{if } x > 2 
\end{cases} \)  
(a) \( \lim_{x \to 2^+} f(x) \)  
(b) \( \lim_{x \to 2^-} f(x) \)  
(c) \( \lim_{x \to 2} f(x) \)
In *Exercise 11*, evaluate the limits if they exist in parts (a) to (k) from the graph of the function \( f \) sketched below.

11. The domain of \( f \) is \([-1, 5]\).

\[
\begin{align*}
(a) \quad & \lim_{{x \to -1}} f(x) \\
(b) \quad & \lim_{{x \to 0}} f(x) \\
(c) \quad & \lim_{{x \to 0}} f(x) \\
(d) \quad & \lim_{{x \to 2}} f(x) \\
(e) \quad & \lim_{{x \to 2}} f(x) \\
(f) \quad & \lim_{{x \to 2}} f(x) \\
(g) \quad & \lim_{{x \to 2}} f(x) \\
(h) \quad & \lim_{{x \to 3}} f(x) \\
(i) \quad & \lim_{{x \to 3}} f(x) \\
(j) \quad & \lim_{{x \to 3}} f(x) \\
(k) \quad & \lim_{{x \to 5}} f(x)
\end{align*}
\]

In *Exercise 12*, sketch the graph of some function \( f \) satisfying the given properties.

12. The domain of \( f \) is \([-4, 4]\). \( f(-4) = 3; \ f(-2) = -3; \ f(0) = 1; \ f(2) = -1; \ f(4) = 0; \)

\[
\begin{align*}
\lim_{{x \to -4}} f(x) = 0; \quad & \lim_{{x \to -2}} f(x) = 1; \quad \lim_{{x \to 0}} f(x) = 1; \quad \lim_{{x \to 0}} f(x) = 4; \quad \lim_{{x \to 2}} f(x) = -1; \\
\lim_{{x \to 4}} f(x) = 0
\end{align*}
\]
Exercise 2.4  ONE – SIDED LIMITS

Solve the following problems.

1. Given the split function defined by
\[ f(x) = \begin{cases} 
1 & \text{if } x = 1 \\
-1 & \text{if } x \neq 1 
\end{cases} \]
Sketch the graph and evaluate
\[ \lim_{x \to 1^-} f(x), \quad \lim_{x \to 1^+} f(x) \text{ and } \lim_{x \to 1} f(x) \]

Solution

2. Given the function \( f(x) = \sqrt{4 - x^2}, x \geq 2. \)
Sketch the graph and evaluate
\[ \lim_{x \to 2^-} f(x), \quad \lim_{x \to 2^+} f(x) \text{ and } \lim_{x \to 2} f(x) \]

Solution

3. Given \( f(x) = \begin{cases} 
x^2 & \text{if } x \leq -2 \\
ax + b & \text{if } -2 < x < 2 \\
2x - 6 & \text{if } 2 \leq x 
\end{cases} \). Find the values of \( a \) and \( b \) such that
\[ \lim_{x \to -2^-} f(x) \text{ and } \lim_{x \to 2^+} f(x) \text{ both exist.} \]
In Exercise 4, evaluate the limits if they exist in parts (a) to (k) from the graph of the function $f$ sketched below.

4. The domain of $f$ is $[0, 5]$.

(a) \( \lim_{x \to 0^-} f(x) \)  
(b) \( \lim_{x \to 1} f(x) \)  
(c) \( \lim_{x \to 1^+} f(x) \)  
(d) \( \lim_{x \to 1} f(x) \)  
(e) \( \lim_{x \to 2^-} f(x) \)  
(f) \( \lim_{x \to 2^+} f(x) \)  
(g) \( \lim_{x \to 2} f(x) \)  
(h) \( \lim_{x \to 4^-} f(x) \)  
(i) \( \lim_{x \to 4^+} f(x) \)  
(j) \( \lim_{x \to 4} f(x) \)  
(k) \( \lim_{x \to 5^-} f(x) \)  

In Exercise 5, sketch the graph of some function $f$ satisfying the given properties.

5. The domain of $f$ is $[-1, 3]$. \( f(-1) = -2; \ f(0) = 0; \ f(1) = 2; \ f(2) = 4; \ f(3) = 1; \)

\[
\lim_{x \to -1^+} f(x) = -2; \ \lim_{x \to 0^-} f(x) = 0; \ \lim_{x \to 0^+} f(x) = 3; \ \lim_{x \to 1^-} f(x) = 4; \ \lim_{x \to 2^-} f(x) = 4; \ \\
\lim_{x \to 2^+} f(x) = 0; \ \lim_{x \to 3^-} f(x) = 5
\]
2.5 LESSON 5  CONTINUITY

2.5.1 SPECIFIC OBJECTIVES

At the end of the lesson you should be able to

- sketch the graphs of continuous and discontinuous functions
- define continuity, left- and right- hand continuity, and continuity on closed and half-open interval
- apply the concepts of limits and continuity to solve real world problems

2.5.2 PREREQUISITE SKILLS

- Algebraic computational and graphing skills

2.5.3 TIME FRAME

- 3 HOURS

2.5.4 LESSON PROPER

We noticed in Lessons 1 – 4 that the limit of a function as \( x \) approaches \( a \) can often be found simply by calculating the value of the function at \( a \). Functions with this property are called continuous at \( a \).

We can intuitively understand the concept on continuity of a function by just informally saying that a function is said to be continuous on an interval if there is no interruption in the graph of \( f \) on that given interval, that is we can trace out the graph of the function without lifting the ball pen. This means that the graph of \( f \) has no hole, jump, gap or break. Otherwise, the function is discontinuous and we say that the location of its discontinuity is at the \( x \) – coordinate where we encounter the gap, hole, jump or break of the function.
Analyze the following different graphs of a function with discontinuities and continuity on a given interval as illustrated in Figure 17.

**Figure 17**

(a) continuous on \([a, b]\)

(b) continuous on \((-\infty, \infty)\)

(c) discontinuous at \(x = b\)

(d) discontinuous at \(x = x\)

(e) discontinuous at \(x = a\)

(f) discontinuous at \(x = x\)
We see from Figure 17 that a function may be continuous or discontinuous at a certain \( x \) coordinate \((x = a)\) on an interval. We now discuss the three reasons why a certain function has discontinuity at \( x = a \).

First, consider the functions \( f(x) = \frac{x^2 - 4}{x - 2} \) and \( g(x) = \frac{1}{x^2} \). The graphs of \( f \) and \( g \) are illustrated in Figure 18.

We see from the graphs that \( f \) and \( g \) are discontinuous at \( x = 2 \) and 0, respectively. We now say that a function is discontinuous if \( f \) is not defined for some \( x \) value.

Second, study the split function \( f(x) = \begin{cases} x^2 + 1 \quad \text{if} \quad x < 1 \\ -x + 1 \quad \text{if} \quad x \geq 1 \end{cases} \). Figure 19 shows the graph of \( f \).

What do you notice on the graph? The limit as \( x \) approaches 1 does not exist. This is because \( \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x) \).

That is, the left-hand limit and the right-hand limits are not equal.
Finally, we demonstrate the third reason by the split function

$$f(x) = \begin{cases} \frac{3}{2}(x + 2) & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

We can see from Figure 20 that $f(2) = 3$ and $\lim_{x \to 2} f(x) = \lim_{x \to 2} f(x) = 6$. Thus, the third reason for discontinuity of a function at $x = a$ is that $\lim_{x \to a} f(x) \neq f(a)$.

We are now ready to use these three reasons to formally define continuity of a function.

**Definition** A function $f$ is **continuous at a number $a$** if

$$\lim_{x \to a} f(x) = f(a)$$

If it is not continuous at $a$, we say that $f$ is **discontinuous at $a$**, or $f$ has a **discontinuity at $a$**. Notice that the definition implicitly requires three things if $f$ is continuous at $a$:

1. $f(a)$ is defined (that is, $a$ is the domain of $f$)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If one or more of this conditions are not satisfied, we say that the function is discontinuous at $x = a$. 

*Figure 20*
Example 1  Discuss the continuity of \( f(x) = 2 - x^3 \) at \( x = 1 \).

**SOLUTION**  We apply the definition of the continuity to prove that of \( f(x) = 2 - x^3 \) is continuous at \( x = 1 \). We have

i. \( f(1) = 2 - (1)^3 = 2 - 1 = 1 \)

ii. \( \lim_{x \to 1} (2 - x^3) = 2 - (1)^3 = 1 \)

iii. \( \lim_{x \to 1} (2 - x^3) = f(1) \)

Thus, \( f(x) = 2 - x^3 \) is continuous at \( x = 1 \) (see Figure 21).

Example 2  Is \( f(x) = \frac{x^2 - 1}{x - 1} \) continuous at \( x = 1 \)?

**SOLUTION**  The function is undefined if the denominator \( x - 1 \) is zero, that is if \( x = 1 \).

To prove that this function is discontinuous at \( x = 1 \), using the definition of continuity (see Figure 22), we get

i. \( f(1) \) is undefined since 1 is not in the domain of \( f \)

ii. \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1} x + 1 = 1 + 1 = 2 \)

iii. \( \lim_{x \to 1} f(x) \neq f(1) \)

**Definition of Right-Hand Continuity**

The function \( f \) is said to be **continuous from the right at the number** \( a \) if and only if the following three conditions are satisfied:

1. \( f(a) \) exists;
2. \( \lim_{x \to a^+} f(x) \) exists;
3. \( \lim_{x \to a^+} f(x) = f(a) \)

Example 3  The function \( g \) is defined by \( g(x) = \sqrt{9 - x^2} \). Prove that this function is continuous on the closed interval \([-3, 3]\).

**SOLUTION**  The function \( g \) is
Thus, \( g \) is continuous from the right at \( x = -3 \) and continuous from the left at \( x = 3 \). Hence, by the Definition of Continuity on a Closed Interval, \( g \) is continuous at every number \( x \) for which \( 9 - x^2 > 0 \). Hence, \( g \) is continuous at every number in an open interval \((-3, 3)\). Evaluating \( \lim_{x \to -3^+} g(x) \) and \( \lim_{x \to 3^-} g(x) \), we have

\[
\lim_{x \to -3^+} g(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = 0 = g(-3)
\]

and,

\[
\lim_{x \to 3^-} g(x) = \lim_{x \to 3^+} \sqrt{9 - x^2} = 0 = g(3)
\]
Example 4  Determine the largest interval (or union of intervals) on which the following function is continuous:

\[ f(x) = \frac{\sqrt{9-x^2}}{x-2} \]

SOLUTION  We first determine the domain of \( f \). The function is defined everywhere except when \( x = 2 \) or when \( 9 - x^2 < 0 \), i.e., when \( x > 3 \) or \( x < -3 \). Therefore, the domain of \( f \) is \([-3, 2) \cup (2, 3]\). Because

\[
\lim_{x \to -3^+} f(x) = 0 \quad \text{and} \quad \lim_{x \to 3^-} f(x) = 0
\]

\[ = f(-3) \quad \text{and} \quad = f(3) \]

\( f \) is continuous from the right at \( x = -3 \) and from the left at \( x = 3 \). Furthermore, \( f \) is continuous on the open intervals \((-3, 2) \) and \((2, 3)\). Therefore, \( f \) is continuous on \([-3, 2) \) and \((2, 3)\).  (refer to Figure 24)
Discuss the continuity at the indicated value of $a$ and sketch the graph of $y = f(x)$.

1. $f(x) = x^2 - 4x + 4$, $a = 2$

SOLUTION

2. $f(x) = \frac{1}{x + 2}$, $a = -2$

SOLUTION

3. $f(x) = \begin{cases} 
  x + 2 & \text{if } x < 1 \\
  2 & \text{if } x = 1 \\
  2 - x & \text{if } x > 1 
\end{cases}$

SOLUTION
In exercises 1 through 5, sketch the graph of the function. By observing where there is a break or gap, determine the number at which the function is discontinuous, and explain why the definition of continuous is not satisfied at this number.

1. \( f(x) = \frac{x^2 + x - 6}{x + 3} \)

2. \( g(x) = \frac{x^2 - 3x - 4}{x - 4} \)

3. \( f(x) = \begin{cases} \frac{x^2 - 5x + 6}{x - 2} & \text{if } x \neq 2 \\ -1 & \text{if } x = 2 \end{cases} \)

4. \( f(x) = \frac{|x|}{x} \)

5. \( f(x) = \frac{x - 9}{\sqrt{x - 3}} \)

6. Suppose at \( t \) meters, \( r(t) \) meters is the radius of a circular oil spill from a ruptured tanker and \( r(t) = \begin{cases} 4t^2 + 20 & \text{if } 0 \leq t \leq 2 \\ 16t + 4 & \text{if } t > 2 \end{cases} \).

   Prove that \( r \) is continuous at 2.

7. According to Einstein’s Theory of Relativity, no particle with positive mass can travel faster than the speed of light. The theory specifies that if \( m(v) \) is the measure of the mass of a particle moving with a velocity of measure \( v \), then
\[ m(v) = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \]

where \( m_0 \) is the constant measure of the particle’s rest mass relative to some reference frame, and \( c \) is the constant measure of the speed of light. Determine the largest interval on which \( m \) is continuous.

In Exercises 8 through 10, sketch the graph of the function \( f \) that satisfies the given properties.

8. \( f \) is continuous on \((-\infty, 2] \) and \((2, +\infty)\); \( \lim_{x \to 0^+} f(x) = 4; \lim_{x \to 2^-} f(x) = -3; \lim_{x \to 2^+} f(x) = +\infty; \lim_{x \to 5} f(x) = 0 \)

9. \( f \) is continuous on \((-\infty, -3], (-3, 3), \) and \([3, +\infty)\); \( \lim_{x \to -5} f(x) = 2; \lim_{x \to -3} f(x) = 0; \lim_{x \to -3^+} f(x) = 4; \lim_{x \to 0^-} f(x) = 1; \lim_{x \to 0^+} f(x) = 0; \lim_{x \to 3^-} f(x) = 0; \lim_{x \to 3^+} f(x) = -5; \lim_{x \to 4} f(x) = 0 \)

10. \( f \) is continuous on \((-\infty, 0) \) and \([0, +\infty)\); \( \lim_{x \to -4} f(x) = 0; \lim_{x \to 0^+} f(x) = 3; \lim_{x \to 0^-} f(x) = -3; \lim_{x \to 4^-} f(x) = 2 \)
REFERENCES


Answers to Selected Odd-Numbered Self – Test and Exercise Problems

Self – Test 2.1

1. \[ \lim_{x \to 4} f(x) = 7 \]

5. \[ \lim_{x \to -2} f(x) = -4 \]

3. \[ \lim_{x \to 3} f(x) = \lim_{x \to 3}(2x + 1) = 7 \]

Exercise 2.1

1. 

<table>
<thead>
<tr>
<th>x</th>
<th>3.00</th>
<th>3.50</th>
<th>3.90</th>
<th>3.99</th>
<th>3.999</th>
<th>5.00</th>
<th>4.50</th>
<th>4.10</th>
<th>4.01</th>
<th>4.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-1</td>
<td>-0.5</td>
<td>-0.1</td>
<td>-0.01</td>
<td>-0.001</td>
<td>1</td>
<td>0.5</td>
<td>0.1</td>
<td>0.01</td>
<td>0.001</td>
</tr>
</tbody>
</table>
3.  

<table>
<thead>
<tr>
<th>x</th>
<th>3.00</th>
<th>2.50</th>
<th>2.10</th>
<th>2.01</th>
<th>2.001</th>
<th>2.00</th>
<th>1.99</th>
<th>1.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-10</td>
<td>-8.5</td>
<td>-7.3</td>
<td>-7.03</td>
<td>-7.003</td>
<td>-4</td>
<td>-5.5</td>
<td>-6.7</td>
</tr>
</tbody>
</table>

\[
\lim_{x \to 4} f(x) = 0
\]

\[
\lim_{x \to -2} f(x) = -7
\]

Self Test 2.3.A

1.  -2
3.  -7
5.  -1
7.  49
9.  -8
11. 0
13. -5341

Self Test 2.3.B

1.  0
3.  -2
5.  -2
7.  \( \frac{1}{3} \)
9.  \( \frac{1}{3} \)
11. 0
13. -2
15. 0
Exercise 2.3

1. 22
2. 27

Self – Test 2.4

1. \( \lim_{x \to 1} (3x + 2) = -1 \), \( \lim_{x \to 1} (x + 2) = -1 \)
3. \( \lim_{x \to 1} \left[ \frac{x^2 + 2x + 1}{x + 1} \right] = 0 \), \( \lim_{x \to 1} \left[ \frac{x^2 + 2x + 1}{x + 1} \right] = 0 \)
5. \( \lim_{x \to \infty} (x\sqrt{x} + 1) = 6 \), \( \lim_{x \to \infty} (x\sqrt{x} + 1) = 6 \)

7. a. \( \lim_{t \to -4} f(t) = 8 \)
   b. \( \lim_{t \to -4} f(t) = 0 \)
   c. \( \lim_{t \to -4} f(t) \) does not exist

9. a. \( \lim_{x \to 5} f(x) = 0 \)
   b. \( \lim_{x \to 5} f(x) = 0 \)
   c. \( \lim_{x \to 5} f(x) = 0 \)

11. a. 0   b. 3   c. 0   d. dne
    e. 5   f. 5   g. 5   h. 2
    i. 2   j. 2   k. 0
Exercise 2.4

1. a. \( \lim_{x \to 1} f(x) = -1 \)
   
   b. \( \lim_{x \to 1} f(x) = -1 \)
   
   c. \( \lim_{x \to 1} f(x) = -1 \)

3. \( a = \frac{3}{2}, \quad b = 1 \)

Self – Test 2.5

1. \( f \) is continuous for all \( x \in (\infty, \infty) \)

3. Discontinuous at \( x = 1 \), since
   \[
   \lim_{x \to 1^-} f(x) = 3, \quad \lim_{x \to 1^+} f(x) = 1 \quad \text{and} \quad f(1)=2
   \]
Exercise 2.5

1. Discontinuous at $x = -3$, since $f(-3)$ is indeterminate and $\lim_{x \to -3} f(x) = -5$

3. $f$ is continuous for all $x \in (\infty, \infty)$

5. $f$ is continuous on $x \in [0,9) \cup (9, \infty)$, since $\lim_{x \to 0^+} f(x)$ does not exist and $f(9)$ is indeterminate.

7. $m$ is continuous on $(-c, c)$

9.